

Explicit cancellation of triangles in one-loop gravity amplitudes

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ABSTRACT: We analyse one-loop graviton amplitudes in the field theory limit of a genus-one string theory computation. The considered amplitudes can be dimensionally reduced to lower dimensions preserving maximal supersymmetry. The particular case of the one-loop five-graviton amplitude is worked out in detail and explicitly features no triangle contributions. Based on a recursive form of the one-loop amplitude we investigate the contributions that will occur at n -point order in relation to the “no-triangle” hypothesis of $\mathcal{N} = 8$ supergravity. We argue that the origin of unexpected cancellations observed in gravity scattering amplitudes is linked to general coordinate invariance of the gravitational action and the summation over all orderings of external legs. Such cancellations are instrumental in the extraordinary good ultra-violet behaviour of $\mathcal{N} = 8$ supergravity amplitudes and will play a central role in improving the high-energy behaviour of gravity amplitudes at more than one loop.

KEYWORDS: Supersymmetric Effective Theories, Supersymmetric gauge theory.

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1. Introduction

Explicit evaluation of graviton scattering amplitudes is a complex and difficult subject using traditional Feynman diagram techniques. Amplitudes for trees and loops with unspecified external polarisation tensors tend to be rather unmanageable, to hide manifest symmetries and to exhibit undesirable features such as a factorial increase in complexity with the number of external legs. This makes the current knowledge of perturbative scattering amplitudes for gravity limited and to a large degree based on assumptions from power counting arguments rather than explicit calculations. In the context of four dimensional maximal supergravity power counting arguments indicate possible ultra-violet divergences at three loops [1, 2], at seven loops [3], at eight loops [4, 5] or at nine loops [6] depending on the implemented superspace formalism. But so far no divergences have been found in explicit calculations [7].

To avoid the myriad of tensor contractions and to generally simplify calculations associated with a conventional field theory approach, string theory can be used as a guideline for calculations. Expressions for field theory amplitudes preserving supersymmetry can be derived in the infinite tension limit ($\alpha' \rightarrow 0$) of the string. String theory rules for graviton amplitudes that hold at tree level have been formulated very elegantly by Kawai, Lewellen and Tye [8] and in [9]. Graviton amplitudes at tree level from string theory was also investigated in [10]. Interestingly such rules also hold in a number of different scenarios [11, 12]

with various matter contents [13]. At one-loop level string based rules have been formulated for amplitude calculations in both gauge theory and gravity [14, 15]. This paper investigates perturbative scattering amplitudes for gravitons in maximal supergravity at one-loop using string theory based techniques relying on the RNS formalism [16–18]. In this work we will consider the conventional field theory limit of a one-loop string amplitude and we will not be affected by the issue raised in [19].

In D dimensions due to the two derivative coupling nature of gravitational interactions an n -graviton amplitude at one loop in gravity has the mass dimension

$$[\mathcal{M}_n] \sim mass^D \tag{1.1}$$

The dimensionful coupling of graviton amplitudes will render gravity inherently non-renormalisable and the n -point one-loop pure graviton amplitude is naïvely given by a Feynman integral with $2n$ powers of loop momenta in the numerator

$$\mathcal{M}_n \sim \int d^D \ell \frac{\prod_{j=1}^{2n} \ell \cdot q_j}{\prod_{i=1}^n (\ell - k_{1\dots i})^2} \tag{1.2}$$

Here $k_{1\dots i} = k_1 + \dots + k_i$, and in the numerator q_j represents some (linear combination) of the external momenta. Using this count for amplitudes in maximally supergravity [20] the maximum number of loop momenta expected to be in the numerator in this case is reduced by eight by supersymmetry. This leads to an overall total number of $2n - 8$ powers of loop momenta in the numerator. This mean for $\mathcal{N} = 8$ graviton amplitudes in $D = 4$ that we are expected to observe triangle integral functions at five points and both triangle and bubble integral functions for amplitudes with six external legs. For seven and higher point amplitudes besides triangle and bubble integral functions - non-analytic rational contributions should be present as well in the amplitude.

Recently, initiated by a paper by Witten [21], there has been new explicit calculations of scattering amplitudes, both for gauge theories (for a review see [22, 23]) and gravity. These new results are to a large extend based on the spinor-helicity [24] formalism in $D = 4$. This has led to new information about scattering amplitudes for gravity and has allowed power counting estimates for graviton amplitudes to be tested by explicit computation. In the theory of maximal supergravity it has been observed in a number of concrete amplitude computations [25–29] that one-loop amplitudes exhibits mysterious unexpected simplifications. These simplifications renders the integral functions in gravity closer to Yang-Mills theory than would otherwise be expected from the naïve counting that was presented above. This has also been referred to as the “no-triangle” hypothesis of $\mathcal{N} = 8$ supergravity [28, 29].

The concept of unexpected simplifications is also supported in a number of recent string theory computations [6, 30] where important input from the pure spinor formalism of Berkovits [31] and string theory dualities points towards a much better UV-behaviour for gravity amplitudes than one should expect from power counting in supersymmetry alone.

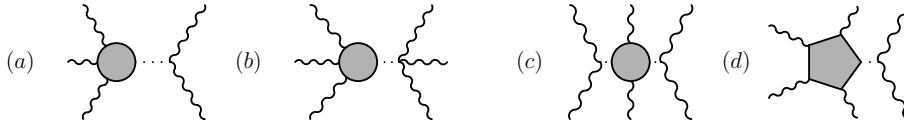


Figure 1: Contribution from one-particle reducible graphs. Up to and including five-graviton amplitudes the reducible graphs are only constructed from boxes, but for six-graviton and beyond higher point amplitudes can occur in the reducible part of the amplitude.

In concrete computations of $\mathcal{N} = 8$ supergravity amplitudes at most $n - 4$ powers of loop momenta appear to be present in the numerator of a generic one-loop amplitude

$$\mathcal{M}_n \sim \int d^D \ell \frac{\prod_{j=1}^{n-4} \ell \cdot q_j}{\prod_{i=1}^n (\ell - k_{1\dots i})^2} \quad (1.3)$$

This suggests that a one-loop n -graviton $\mathcal{N} = 8$ supergravity amplitude can be reduced to a sum of massive box integrals multiplied by an operator of mass dimension eight and n -gons, *i.e.*, ($n \geq 5$) scalar integrals evaluated in dimensions $D + 2k$ with $0 \leq k \leq n - 4$ [25, 28, 32, 29] for $D > 4$. In $D = 4$ the no-triangle hypothesis suggest that one-loop n -graviton amplitudes in $\mathcal{N} = 8$ do not contain integral functions more singular than (massive) boxes and in particular do not contain triangles nor bubble functions. Including the unexpected cancellations one will have in the generic case for an arbitrary supersymmetric theory [29]

$$\nu \leq 2n - (n - 4 + \mathcal{N}) = n + 4 - \mathcal{N} \quad (1.4)$$

ν loop momenta in the numerator for theories with $0 \leq \mathcal{N} \leq 8$ supersymmetries (replace \mathcal{N} by $\mathcal{N} + 1$ for an odd number of supersymmetries in (1.4)).

The no-triangle hypothesis does carry cancellations into multi-loop amplitudes. This can be observed through cuts of amplitudes and through physical factorisation limits linking $m < n$ loop amplitudes to n -loop amplitudes [33]. At multi-loop level it has been verified that $\mathcal{N} = 8$ supergravity is a finite theory until three loops [7].

In deriving the field theory limit of the n -gravitons amplitude at genus one in string theory from the contributions of colliding vertex operators [34, 35] it is observed that the total contribution to the n -graviton amplitude at one-loop is composed of *one-particle irreducible* contributions and *one-particle reducible* contributions, see figure 1. These contributions originates from the boundary of the moduli of the punctured Riemann surface on which the string amplitude is defined [16].

The one-particle reducible contributions, displayed in figure 1(b) arise from the possibility of constructing a one-loop amplitude by attaching k -point tree vertices to a one-loop $n - k$ -point amplitude in order to construct n -point contributions.

It is a well known fact that linearised $\mathcal{N} = 8$ supersymmetry of perturbative string theory guaranties that the one-, two-, and three-point amplitudes are vanishing at one-loop. Therefore in $\mathcal{N} = 8$ supergravity (and type II superstring theory) there is no room for constructing reducible graphs from triangles or bubbles. This fact was noticed in [5].

This however does not imply the absence of triangles in maximal supergravity gravity amplitudes because supersymmetry allows higher-point reducible and irreducible amplitudes contributions that contain triangles. At one-loop order, the supersymmetric cancellations enforced by the saturation of the sixteen fermionic zero modes only subtract eight powers of loop momenta leading to contributions of the type (again we display only the contributions with the most powers of loop momentum)

$$\mathcal{M}_n^{1PI} \sim \mathcal{O}_8 \int d^D \ell \frac{\prod_{j=1}^{2(n-4)} \ell \cdot q_j}{\prod_{i=1}^n (\ell - k_{1\dots i})^2} \tag{1.5}$$

where \mathcal{O}_8 is a mass dimension eight operator factorising in front of the loop amplitude. Hence we obtain triangle contributions after $(n - 3)$ steps of Passarino-Veltman reductions. No known explanation for cancellations of triangles has been attributed solely to supersymmetry.

In this paper we will consider the explicit computation of the five-graviton amplitude at one-loop in maximal supergravity. The five-graviton one-loop MHV amplitude in four dimensions has already been derived using the on-shell unitarity methods in [36]. The method used in the present paper is different and is not restricted to a particular dimension. A direct comparison with the results of that paper will appear in [37]. We will use the form of the n -graviton amplitude provided by the field theory limit of the genus-one amplitude compactified on a torus. String theory allows us to implement in a simple way the effects of the $\mathcal{N} = 8$ supersymmetry by using the Jacobi identity (and its generalisation for higher-point amplitudes). This provides a practical set-up for classifying the reducible contributions and enables us to recursively construct the n -point amplitude. We discuss the contributions that will occur in higher-point amplitudes.

2. The five-graviton amplitude

We will in this section consider the derivation of the five-point amplitude in maximal supergravity in D dimensions from the field theory limit of type II string theory compactified on a $10 - d$ dimensional torus. In order to derive the five-graviton one-loop amplitude we will use the rules of perturbative string theory at genus one. A basic presentation of the employed string theory rules and a discussion of the field theory limit are offered in the appendices. Further details will appear in [37]. The resulting field theory amplitude is given by an irreducible contribution and a reducible contribution displayed below in figure 2(a) and 2(b) respectively. We will analyse these contributions in turn.

2.1 The one-particle irreducible contributions

The one-particle irreducible contribution to the five-graviton amplitude at one-loop receives contributions both from the *even/even* spin structure sector of the genus-one string theory amplitude and from the *odd/odd* spin structure sector of the amplitude.



Figure 2: Contribution to the field theory limit of the five-graviton amplitude.

Working out the bosonic contractions, the integrand of the five-point amplitude in the *even/even* spin structure sector takes the form

$$\begin{aligned}
 A_5^{e/e} = & \frac{\kappa_{(10)}^2}{\alpha'^{\frac{D}{2}-5}} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \Gamma_{(10-D,10-D)} \prod_{i=1}^4 \int_{\mathcal{T}} \frac{d^2z_i}{\tau_2} \prod_{1 \leq i < j \leq 5} |\chi(z_i - z_j)|^{-\alpha' k_i \cdot k_j} \times \quad (2.1) \\
 & \times \left(\left| T_{10} \cdot F^5 + i\pi \sum_{i,j=1}^5 k_i \cdot k_j \partial_i \ln \chi(z_i - z_j) (t_8 \cdot F_i^4) \right|^2 \right. \\
 & \left. - \sum_{i \neq j} \frac{k_i \cdot k_j}{\alpha'} \partial_i \bar{\partial}_j \ln \chi(z_i - z_j) (t_8 \cdot F_i^4) (t_8 \cdot F_j^4) \right)
 \end{aligned}$$

The integrations in the above formula are over the positions of the external states $z_i = \nu_i^{(1)} + i\tau_2 \nu_i$ where the domain of integration is $\mathcal{T} = \{|\nu^{(1)}| \leq 1/2, \nu \in [0, 1]\}$ and $z_5 = \tau$ by conformal invariance. The factor $\Gamma_{(10-D,10-D)}$ represents the contributions from the winding modes and Kaluza-Klein states. Here $T_{10} \cdot F^5$ is the contribution from the contraction of the ten world-sheet fermions defined in eq. (A.9). The quantity $t_8 \cdot F_i^4$ is defined as the contractions of the field strengths $F_{\mu\nu} = h_\mu k_\nu - h_\nu k_\mu$ (of the four states different from state i) with the usual $t_8^{\mu_1 \nu_1 \dots \mu_4 \nu_4}$ tensor defined in appendix 9.A of [17]. The bosonic propagators are given by $\ln \chi(z)$ defined in eq. (A.6) of the appendix. We refer to the appendix for our conventions and for a further discussion of the field theory limit.

The 1PI contributions to the field theory limit of the amplitude will be obtained in the limit of $\alpha' \rightarrow 0$ and $\tau_2 \rightarrow \infty$ while keeping $t = \alpha' \tau_2$ and the distance between the vertex operators finite. In this limit the fermionic and bosonic propagators are

$$\begin{aligned}
 S_1(z) & \rightarrow G_F(\nu) = \pi \operatorname{sign}(\nu) \\
 \partial_z \ln \chi(z) & \rightarrow \dot{G}_B(\nu) = \pi \nu - \frac{1}{2} G_F(\nu) \\
 \partial_z \bar{\partial}_z \ln \chi(z) & \rightarrow -\alpha' \frac{\pi}{4} \frac{1}{t} \quad (2.2)
 \end{aligned}$$

Taking $R \rightarrow 0$ and $\alpha'/R \rightarrow 0$ the lattice sum has the limit

$$\Gamma_{(10-D,10-D)} \rightarrow R^{5-\frac{D}{2}} \tau_2^{5-\frac{D}{2}} \quad (2.3)$$

We introduce the n -point integrals

$$I_n^{(D)}[f(\nu)] \equiv \pi^{\frac{D}{2}-n} \Gamma\left(n - \frac{D}{2}\right) \prod_{i=1}^n \int_0^1 d\nu_i f(\nu_i) Q_n(k_i)^{\frac{D}{2}-n} \delta(\nu_n - 1) \quad (2.4)$$

where

$$Q_n(k_i) = \sum_{1 \leq i < j \leq n} (k_i \cdot k_j) \left[(\nu_i - \nu_j)^2 - |\nu_i - \nu_j| \right] \quad (2.5)$$

The 1PI contribution to the five-graviton genus-one amplitude leads to the result

$$\mathcal{M}_5^{1PI} = I_5^{(D)} \left[|\mathcal{A}_5^{(1)\infty}|^2 \right] + \pi I_5^{(D+2)} \left[\mathcal{A}_5^{(2)\infty} \right] \quad (2.6)$$

This expression assumes the summation of all the ordering of the external legs. We refer to appendix A.2 for further details. The second term has a dimension shift from D to $D + 2$ which arises from the extra inverse power of the loop proper time from the zero mode contribution of the bosonic coordinates.

The various pieces of the field theory amplitudes are given by

$$\begin{aligned} \mathcal{A}_5^{(1)\infty} &= t_{10} \cdot F^5 + \pi \sum_{i \neq j} (h_i \cdot k_j) \dot{G}_B(\nu_i - \nu_j) (t_8 \cdot F_i^4) \\ &= t_{10} \cdot F^5 - \frac{\pi}{2} \sum_{i \neq j} (h_i \cdot k_j) G_F(\nu_i - \nu_j) (t_8 \cdot F_i^4) - \pi H \cdot K_{[5]} \\ \mathcal{A}_5^{(2)\infty} &= \sum_{i \neq j} h_i \cdot \bar{h}_j (t_8 \cdot F_i^4) (t_8 \cdot F_j^4) \end{aligned} \quad (2.7)$$

Here H and \bar{H} defined as

$$H = \sum_{i=1}^5 h_i (t_8 \cdot F_i^4), \quad \bar{H} = \sum_{i=1}^5 \bar{h}_i (t_8 \cdot \bar{F}_i^4) \quad (2.8)$$

have been introduced together with

$$K_{[n]} = \sum_{i=1}^n k_i \nu_i \quad (2.9)$$

The quantity $t_{10} \cdot F^5$ (defined in eq. (A.18)) depends on the ordering of the positions of the vertex operators. It is defined as the field theory limit of the contractions between the fermions $T_{10} \cdot F^5$ in the string theory amplitude.

From the contribution $\mathcal{A}_5^{(1)\infty}$ one gets a combination of scalar pentagons $M_5[1]$, a combination of linear pentagons $M_5[\nu]$ and a linear combination of quadratic pentagons $M_5[\nu^2]$

$$M_5[1] = I_5^{(D)} \left[\left| t_{10} \cdot F^5 - \frac{\pi}{2} \sum_{i \neq j} (h_i \cdot k_j) G_F(\nu_i - \nu_j) (t_8 \cdot F_i^4) \right|^2 \right] \quad (2.10)$$

$$M_5[\nu] = -\pi I_5^{(D)} \left[\left(t_{10} \cdot F^5 - \frac{\pi}{2} \sum_{i \neq j} (k_i \cdot h_j) G_F(\nu_i - \nu_j) (t_8 \cdot F_i^4) \right) (H \cdot K_{[5]}) \right] \quad (2.11)$$

$$M_5[\nu^2] = \pi^2 I_5^{(D)} \left[(H \cdot K_{[5]}) (\bar{H} \cdot K_{[5]}) \right] \quad (2.12)$$

The expressions given here are summed over all the ordering of the external legs. For evaluating these expressions and extracting the various contributions having branch cuts in different kinematic channels, one has to split the integral with respect to the various orderings; see [14, 38, 39] and the appendix.

The contribution from $\mathcal{A}_5^{(2)\infty}$ in eq. (2.6) has an extra power of Q_5 from the zero mode contribution of the bosonic coordinates and contributes to a linear combination of scalar pentagons in $D + 2$ dimensions

$$M_5^{(D+2)}[1] = \sum_{i \neq j} (h_i \cdot \bar{h}_j) (t_8 \cdot F_i^4) (t_8 \cdot F_j^4) I_5^{(D+2)}[1] \quad (2.13)$$

The *odd/odd* spin structure contribution to a toroidal compactification of the n -graviton amplitude at one-loop vanishes in $D < 10$. This is because of the impossibility of saturating the fermionic zero modes along the compactified directions with only external states without polarisations along the internal directions. The *odd/odd* spin structure contributes to the amplitude in ten dimensions for $n \geq 5$ graviton amplitudes. Its contribution have the following form [40]

$$\mathcal{M}_5^{o/o} \propto \epsilon^{\lambda\mu_1 \dots \mu_9} \epsilon^{\lambda\nu_1 \dots \nu_9} \prod_{1 \leq r \leq 5} h_{\mu_r}^{(r)} \bar{h}_{\nu_r}^{(r)} \prod_{1 \leq s \leq 4} k_{\mu_{5+s}}^{(s)} k_{\nu_{5+s}}^{(s)} I_5^{(12)}[1] \quad (2.14)$$

Only the quadratic pentagons in eq. (2.12) can contain triangles, and we will show in section 4 how of these contributions cancels explicitly.

2.2 The reducible contribution

We now turn to the 1PR contributions. We shall see that they cannot contribute to triangles at this order since they are only given by massive scalar box contributions represented in figure 2(b).

The reducible expressions arise when two (or more) vertex operators collide in the field theory limit. When $z_i \rightarrow z_j$ the bosonic or fermionic propagator develop a pole

$$\lim_{i \rightarrow j} \partial_i \ln \chi(z_i - z_j) = -\frac{1}{4} \frac{1}{z_i - z_j}, \quad \lim_{i \rightarrow j} S_1(\bar{z}_i - \bar{z}_j) = \frac{1}{\bar{z}_i - \bar{z}_j} \quad (2.15)$$

and the integrand $|\mathcal{A}_5^{(1)}|^2$ in eq. (2.1) can develop a pole when, say $z_4 \rightarrow z_5$

$$\lim_{4 \rightarrow 5} |\mathcal{A}_5^{(1)}|^2 \sim \frac{1}{|z_{45}|^2} |t_{(45)}|^2 \quad (2.16)$$

where

$$t_{(45)} \equiv t_8(F^1 F^2 F^3 F^{45}) - h_5 \cdot k_4 (t_8 \cdot F_5^4) + h_4 \cdot k_5 (t_8 \cdot F_4^4) \quad (2.17)$$

with $F_{\mu\nu}^{45} = (F^4)_\mu{}^\lambda (F^5)_{\lambda\nu}$. The sum of the reducible contributions is given by

$$A_{5,1PR}^{e/e} = \frac{k_{(10)}^2}{\alpha'^{\frac{D}{2}-5}} \sum_{i \neq j} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \Gamma_{(10-D, 10-D)} \quad (2.18)$$

$$\prod_{r=1}^3 \int_{\mathcal{I}} \frac{d^2 z_r}{\tau_2} \frac{|t_{(ij)}|^2}{|z_i - z_j|^{2+2\alpha' k_i \cdot k_j}} \prod_{1 \leq u < v \leq 4} |\chi(z_u - z_v)|^{-2\alpha' P_v \cdot P_u}$$

with $z_4 = \tau$ and $P_m = k_m$ if $m \neq i$ or $m \neq j$ and $P_i = k_i + k_j$. The integration over $z_i = z_j + \zeta$ with $|\zeta| < \epsilon \ll 1$ gives in the field theory limit [34, 35]

$$\begin{aligned} \lim_{\alpha' \rightarrow 0} \int_{|\zeta| < \epsilon} d^2 \zeta |\zeta|^{-2\alpha' k_i \cdot k_j - 2} &= - \lim_{\alpha' \rightarrow 0} \frac{\epsilon^{\alpha' k_i \cdot k_j}}{\alpha' k_i \cdot k_j} \\ &= - \frac{1}{\alpha' k_i \cdot k_j} \end{aligned} \tag{2.19}$$

Therefore the reducible contribution to the five-point amplitude is given by

$$\mathcal{M}_5^{1PR} = \lim_{\alpha' \rightarrow 0} \kappa_{(D)}^{-2} \mathcal{A}_{5,1PR}^{e/e} = \pi^{\frac{D-8}{2}} \Gamma\left(\frac{8-D}{2}\right) \sum_{i \neq j} t_{(ij)} \prod_{r=1}^4 \int_0^1 d\nu_r Q_4(P_i)^{\frac{D}{2}-4} \delta(\nu_4 - 1) \tag{2.20}$$

which is the sum of contributions from one-mass scalar boxes evaluated for the external momenta (P_1, P_2, P_3, P_4) .

3. Reduction formulas

In this section we will discuss the integral reduction formulas needed to examine the integral contributions of the five-point amplitude. The reduction formulas presented in [41, 42] could have been used in this analysis, however we found it useful to derive the reduction formulas (which have their root in gauge invariance and the decoupling of longitudinal modes) from the viewpoint of string theory.

We will consider special expressions at genus one between very specific vertex operators. The vertex operators are not describing physical external states but are part of the physical vertex operators of string theory. The identity we will derive in this section will be entering the analysis of the graviton five-point amplitude.

We introduce the fermionic vertex operators (see the appendix for definitions and conventions)

$$V_{\psi\bar{\psi}}^{IJ,KL}(k) = \int_{\mathcal{T}} d^2 z : \psi^I \psi^J \bar{\psi}^K \bar{\psi}^L e^{ik \cdot x(z)} : \tag{3.1}$$

the mixed fermionic and bosonic vertex operators with a longitudinal part

$$\begin{aligned} V_{\partial x \bar{\psi}}^{KL}(k) &= \int_{\mathcal{T}} d^2 z : \partial(\bar{\psi}^K \bar{\psi}^L e^{ik \cdot x(z)}) : \\ &= \int_{\mathcal{T}} d^2 z : ik \cdot \partial x \bar{\psi}^K \bar{\psi}^L e^{ik \cdot x(z)} : \end{aligned} \tag{3.2}$$

$$\begin{aligned} V_{\psi \bar{\partial} x}^{IJ}(k) &= \int_{\mathcal{T}} d^2 z : \bar{\partial}(\psi^I \psi^J e^{ik \cdot x(z)}) : \\ &= \int_{\mathcal{T}} d^2 z : ik \cdot \bar{\partial} X \psi^I \psi^J e^{ik \cdot x(z)} : \end{aligned} \tag{3.3}$$

These expressions are vanishing because the left-moving or the right-moving part of these vertex operators is purely longitudinal and the torus $\mathcal{T} = \{z = \nu^{(1)} + i\tau_2 \nu; |\nu^{(1)}| \leq$

$1/2; \nu \in [0, 1]$ has no boundaries. We introduce as well the purely longitudinal bosonic vertex operator

$$V_{\partial X \bar{\partial} X}(k) = \int_{\mathcal{T}} d^2 z : \partial \bar{\partial} (e^{ik \cdot x(z)}) : = - \int_{\mathcal{T}} d^2 z : k \cdot \partial x k \cdot \bar{\partial} x e^{ik \cdot x(z)} : \quad (3.4)$$

In these expressions $x^\mu(z)$ and $\psi^\mu(z)$ are the conformal fields of weight 0 and 1/2 of the RNS formulation of perturbative string theory. The index μ runs from 0 to $D \leq 10$. The manipulations in this section will be done using the rule for computing correlators at genus-one order in string theory, but the manipulations here do not require that we are working in the critical dimension $D = 10$ neither that we are working with physical vertex operators. In this section the lattice factor $\Gamma_{(10-D, 10-D)}$ has been replaced by its field theory approximation $\tau_2^{5-D/2}$ of eq. (2.3). This scheme was already used in the so-called ‘string based rules’ of [15].

We introduce the following notation

$$\mathcal{O}_{\psi^{2n}, \bar{\psi}^{2n}}(k_{i_1}, \dots, k_{i_n}) = t_{I_{i_1} \dots I_{i_n}}^L t_{J_{i_1} \dots J_{i_n}}^R V_{\psi \bar{\psi}}^{I_{i_1} I_{i_2}, J_{i_1} J_{i_2}}(k_{i_1}) \dots V_{\psi \bar{\psi}}^{I_{i_{n-1}} I_{i_n}, J_{i_{n-1}} J_{i_n}}(k_{i_n}) \quad (3.5)$$

where $t_{I_1 \dots I_n}^L$ and $t_{I_1 \dots I_n}^R$ are rank n -tensors contracting the Lorentz indices of the left moving fermions $\psi^\mu(z)$ and the right moving fermions $\bar{\psi}^\mu(z)$.

We will start by considering the genus one expression involving four fermionic operators evaluated in the *even/even* spin structure sector. The result is

$$\langle \mathcal{O}_{\psi^s, \bar{\psi}^s}(k_1, \dots, k_4) \rangle_{e/e} = t_8 t^L t_8 t^R \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2} \tau_2^{4-\frac{D}{2}} \prod_{i=1}^3 \int_{\mathcal{T}} \frac{d^2 z_i}{\tau_2} \prod_{1 \leq i < j \leq 4} |\chi(z_i - z_j)|^{-\alpha' k_i \cdot k_j} \quad (3.6)$$

with $z_4 = 1$. This result is proportional to the genus-one four-point amplitude in type II superstring which has the field theory limit $\alpha' \rightarrow 0$ in $D = d - 2\epsilon$ dimensions. The one-loop four point scalar box $I_4^{(D)}[1]$ (in the dimensional regularisation scheme) is summed over all the possible ordering of the external legs [14]

$$\lim_{\alpha' \rightarrow 0} \langle \mathcal{O}_{\psi^s, \bar{\psi}^s}(k_1, \dots, k_4) \rangle_{e/e} = t_8 t^L t_8 t^R I_4^{(D)}[1] \quad (3.7)$$

where $I_4^{(D)}[1]$ is defined in eq. (2.4) and, *e.g.*, $t_8 t^L$ is the contraction of t_8 and t^L . This expression is the sum of the *s*-channel $I_4(s, t)$, *t*-channel $I_4(t, u)$ and *u*-channel $I_4(u, s)$ boxes [14, 38, 39].

We consider now the *even/even* spin structure correlator with the insertion of two longitudinal vertex operators $h_{ij}^5 V_{\partial X \bar{\psi}}^{ij} = 0$ and $h_{ij}^4 V_{\psi \bar{\partial} X}^{ij} = 0$ which we defined in eq. (3.3). Now

$$\begin{aligned} 0 &= h_{ij}^5 h_{kl}^4 \left\langle V_{\partial X \bar{\psi}}^{ij}(k_5) V_{\psi \bar{\partial} X}^{kl}(k_4) \mathcal{O}_{\psi^6, \bar{\psi}^6}(k_1, \dots, k_3) \right\rangle_{e/e} \\ &= \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2} \tau_2^{5-\frac{D}{2}} \prod_{i=1}^4 \int_{\mathcal{T}} \frac{d^2 z_i}{\tau_2} \prod_{1 \leq i < j \leq 5} t_8 (h^5 t^L) t_8 (h^4 t^R) |\chi(z_i - z_j)|^{-\alpha' k_i \cdot k_j} \times \end{aligned}$$

$$\times \left[\left(\sum_{j=1}^5 ik_5 \cdot k_j \partial_5 \ln \chi(z_5 - z_j) \right) \left(\sum_{i=1}^5 ik_4 \cdot k_i \bar{\partial}_4 \ln \chi(z_4 - z_i) \right) - \frac{k_5 \cdot k_4}{\alpha'} \partial_5 \bar{\partial}_4 \ln \chi(z_5 - z_4) \right] \quad (3.8)$$

The contractions of the eight left-moving and eight right-moving fermions and the sum over the spin structure have been done using the Jacobi identity given in the appendix. It is important to notice that this gives a contribution that is a constant independent of the positions of the vertex operators. Thus eq. (3.8) implies that

$$0 = \mathcal{R} \equiv \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \tau_2^{5-\frac{D}{2}} \prod_{i=1}^4 \int_{\mathcal{T}} d^2\nu_i \prod_{1 \leq i < j \leq 5} |\chi(z_i - z_j)|^{-\alpha' k_i \cdot k_j} \times \left[\left(\sum_{j=1}^5 ik_5 \cdot k_j \partial_5 \ln \chi(z_5 - z_j) \right) \left(\sum_{i=1}^5 ik_4 \cdot k_i \bar{\partial}_4 \ln \chi(z_4 - z_i) \right) - \frac{k_4 \cdot k_5}{\alpha'} \partial_5 \bar{\partial}_4 \ln \chi(z_5 - z_4) \right] \quad (3.9)$$

In the field theory limit this amplitude gives rise to the one-particle irreducible (1PI) contributions and one-particle reducible contributions (1PR).

The 1PI contribution is obtained using the field theory asymptotic of the bosonic and fermionic propagators given in eq. (2.2). With the same manipulation as for the 1PI contribution to the physical amplitude in eq. (2.1) we obtain

$$\begin{aligned} \mathcal{R}^{1PI} &= -I_5^{(D)} \left[(k_4 \cdot K_{[5]})(k_5 \cdot K_{[5]}) \right] \quad (3.10) \\ &\quad - \frac{1}{2} I_5^{(D)} \left[\left(\sum_{i=1}^5 (k_5 \cdot k_i) \text{sign}(\nu_5 - \nu_i) \right) (k_4 \cdot K_{[5]}) \right] + (4 \leftrightarrow 5) \\ &\quad - \frac{1}{4} I_5^{(D)} \left[\sum_{i,j=1}^5 (k_5 \cdot k_i) \text{sign}(\nu_5 - \nu_i) (k_4 \cdot k_j) \text{sign}(\nu_4 - \nu_j) \right] \\ &\quad - (k_4 \cdot k_5) I_5^{(D+2)} [1] \end{aligned}$$

The expression in eq. (3.10) is the sum of scalar, linear and quadratic pentagons in dimension D and a scalar pentagon in dimension $D + 2$ from the zero mode contribution from the correlator between the bosonic coordinates.

The reducible contributions (1PR) in the field theory limit of \mathcal{R} are obtained only when the bosonic propagators develop a pole as in eq. (2.15) when $z_4 \rightarrow z_5$. Note that when $z_4 \rightarrow z_m$ or $z_5 \rightarrow z_m$ with $m = 1, 2, 3$ the expression (3.9) behaves as $1/(\bar{z}_4 - \bar{z}_m)$ and $1/(z_5 - z_m)$ respectively, which does not lead to a reducible contribution because this requires a $1/|z_i - z_j|^2$ type of singularity as described in eq. (2.19). In this case the expression \mathcal{R} behaves as

$$\mathcal{R}^{1PR} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \tau_2^{5-\frac{D}{2}} \prod_{r=1}^4 \int_{\mathcal{T}} \frac{d^2z_r}{\tau_2} \frac{(k_4 \cdot k_5)^2}{|z_i - z_j|^{2+2\alpha' k_i \cdot k_j}} \prod_{1 \leq u < v \leq 4} |\chi(z_u - z_v)|^{-2\alpha' P_v \cdot P_u} \quad (3.11)$$

with $\{P_m\} = \{k_1, k_2, k_3, k_4 + k_5\}$. Performing the integration over $z_5 = z_4 + \zeta$ with $|\zeta| < \epsilon \ll 1$ as in eq. (2.19) the 1PR contribution to \mathcal{R} is given by the one-mass scalar box obtained by colliding the states 4 and 5

$$\mathcal{R}^{1PR} \equiv \lim_{\alpha' \rightarrow 0} \lim_{4 \rightarrow 5} \mathcal{R} = -(k_4 \cdot k_5) I_4^{(45)}[1] \tag{3.12}$$

Collecting the 1PI and 1PR contributions to the field theory limit of (3.9) gives the following identity

$$\begin{aligned} I_5^{(D)}[(k_4 \cdot K_{[5]})(k_5 \cdot K_{[5]})] &= (k_4 \cdot k_5) I_5^{(D+2)}[1] \\ &+ \frac{1}{2} I_5^{(D)} \left[\left(\sum_{i=1}^5 (k_5 \cdot k_i) \text{sign}(\nu_5 - \nu_i) \right) (k_4 \cdot K_{[5]}) \right] + (4 \leftrightarrow 5) \\ &+ \frac{1}{4} I_5^{(D)} \left[\sum_{i,j=1}^5 (k_5 \cdot k_i) \text{sign}(\nu_5 - \nu_i) (k_4 \cdot k_j) \text{sign}(\nu_4 - \nu_j) \right] \\ &+ (k_4 \cdot k_5) I_4^{(45)}[1] \end{aligned} \tag{3.13}$$

relating a linear combination of quadratic pentagons to a scalar pentagon in $D + 2$ dimensions and scalar and linear pentagons as well as one-mass boxes in D dimensions. The loop integral is defined with the summation over all the orderings and the right-hand-side does not contain any triangles. The same identity is valid for any choice of a pair of momenta k_m and k_n with $(m, n) \in \{1, 2, 3, 4, 5\}^2$. In (3.13) we had $k_m = 4$ and $k_n = 5$.

Similar relations as (3.13) were found in section 6 of [42] using manipulations of Feynman parameter integrals with a fixed ordering of the external legs. Via further reduction of linear pentagons to one-mass boxes it can be observed that $D + 2$ pentagons are not present in the amplitudes in $D = 4$ [42].

4. Cancellation of the triangles

We will now show how the identity given in eq. (3.13) allows us to remove the potential triangle contributions present in the quadratic pentagon $M_5[\nu^2]$ in eq. (2.12) for the five-graviton amplitude in $\mathcal{N} = 8$ supergravity.

For an amplitude with at least five external states there are at least four independent momenta, say k_1, k_2, k_3 and k_4 , in dimension $D \geq 4$. Hence we can decompose H and \bar{H} in such a basis as

$$H = \sum_{i=1}^4 c_i k_i + q_{\perp}, \quad \bar{H} = \sum_{i=1}^4 \bar{c}_i k_i + \bar{q}_{\perp} \tag{4.1}$$

where c_i and \bar{c}_i are constants and q_{\perp} and \bar{q}_{\perp} are orthogonal to the chosen four independent momenta of the external states (this is needed only in $D > 4$). We have assumed a generic configuration of external momenta with no momenta being collinear. The case of collinear momenta is correctly captured by the reduction formulas.

Plugging this decomposition into eq. (2.12) the combination of quadratic pentagons can be rewritten as the linear combination

$$M_5[\nu^2] \propto \sum_{i,j=1}^4 c_i \bar{c}_j I_5^{(D)} \left[(k_i \cdot K_{[5]}) (k_j \cdot K_{[5]}) \right] \quad (4.2)$$

of the same quantities appearing in the left-hand-side of the identity in eq. (3.13). Because the right-hand-side of this identity does not have any triangle contributions, we conclude that the five-graviton amplitude M_5^{1PI} of eq. (2.6) does not contain any triangles.

It should be noted that we have used the reduction formula given in the form of eq. (3.13) directly without having to solve for individual quadratic pentagons. We also note that we did not have to invert the Gram determinant of the external momenta which is very messy at higher-point order because of the linear dependence in the kinematic invariants [41, 42].

5. Cancellation of triangles in higher-point amplitudes

At six-point order the integrand of the amplitude takes the recursive form (see [37] and the appendix)

$$A_6 = T_{12} \cdot F^6 + \sum_i (h_i \cdot \partial X) (T_{10} \cdot F_i^5) + \sum_{i \neq j} (h_i \cdot \partial X) (h_j \cdot \partial X) (t_8 \cdot F_{i,j}^4) \quad (5.1)$$

where the quantity $t_8 \cdot F_{i,j}^4$ is the four point amplitude constructed from the field strengths of the four external states different from i and j .

The reducible graphs are given by the one-mass box of figure 1(b) and the two-mass boxes of figure 1(c) (depending on the ordering of the vertices around the loop this gives the two-mass easy or hard scalar box [36, 25, 28, 32]) and the one-mass pentagon of figure 1(d). Quadratic pentagons in the reducible part of the six-point amplitude can appear from poles arising from the second and the third term in (5.1). In case of a pole from colliding the states 5 and 6 we have the quadratic pentagons

$$\frac{1}{s_{56}} I_5^{[D]} \left[(H^{(6 \rightarrow 5)} \cdot K_{[6]}^{(6 \rightarrow 5)}) (\bar{H}^{(6 \rightarrow 5)} \cdot K_{[6]}^{(6 \rightarrow 5)}) \right] \quad (5.2)$$

Here $K_{[6]}^{(6 \rightarrow 5)}$ is the five-point sum $K_{[5]}$ for the momenta $\{k_1, k_2, k_3, k_4, k_5 + k_6\}$, and

$$H^{(6 \rightarrow 5)} = \sum_{i=1}^4 h_i t_{(56)i} \quad (5.3)$$

is a linear combination of the polarisations weighted by the five-point tensor $t_{(56)i}$ defined as in eq. (2.17) for the external states different from i . Decomposing the tensor (5.3) as a linear combination of k_1, k_2, k_3 and k_4 the analysis of sections 2 and 4 assures that this contribution has no triangles.

So in the six-graviton amplitude the triangle can only be present in the irreducible part. The total amplitude the six-point amplitude contains two types of contributions,

depending on whether there are contractions between left-moving ∂x and right-moving $\bar{\partial} x$ or not. The term involving the left/right contraction are

$$\begin{aligned}
 \mathcal{A}_6^{(2)} &= \sum_{i \neq j} (h_i \cdot \bar{h}_j) \frac{1}{\alpha' \tau_2} (T_{10} \cdot F_i^5) (T_{10} \cdot F_j^5) + \sum_{\substack{i \neq j \\ p \neq q}} (h_i \cdot \bar{h}_j) (h_p \cdot \bar{h}_q) \frac{1}{(\alpha' \tau_2)^2} (t_8 \cdot F_{i,j}^4) (t_8 \cdot F_{p,q}^4) \\
 &+ \sum_{\substack{i \neq j \\ p,q}} (h_i \cdot \bar{h}_j) (\bar{h}_p \cdot k_q) \frac{1}{\alpha' \tau_2} \bar{\partial} \ln \chi(z_p - z_q) (T_{10} \cdot F_i^5) (t_8 \cdot F_{p,j}^4) + c.c. \\
 &+ \sum_{\substack{i \neq j \\ p,q,m,n}} (h_i \cdot \bar{h}_j) (h_p \cdot k_m) (\bar{h}_q \cdot k_n) \frac{1}{\alpha' \tau_2} \partial \ln \chi(z_p - z_m) \bar{\partial} \ln \chi(z_q - z_n) (t_8 \cdot F_{p,i}^4) (t_8 \cdot F_{q,j}^4)
 \end{aligned} \tag{5.4}$$

In the field theory limit this expression leads to 1PI contributions composed by a sum of scalar, linear and quadratic hexagons evaluated in dimension $D + 2$ and a scalar hexagon evaluated in dimension $D + 4$. None of these contributions contain triangles.

The other 1PI contributions to the six-point amplitude can be written as

$$\mathcal{A}_6^{(1)\infty} = \left| t_{12} \cdot F^6 + i \sum_{i,m=1}^6 (h_i \cdot k_m) \dot{G}_B(\nu_i - \nu_m) \mathcal{A}_{5(i)}^{(1)\infty} + \sum_{i \neq j} (h_i \cdot h_j) \frac{\pi}{\alpha' \tau_2} (t_8 \cdot F_{i,j}^4) \right|^2 \tag{5.5}$$

where $\mathcal{A}_{5(i)}^{(1)\infty}$ is the five-point amplitude given in eq. (2.7) evaluated for the five external states different from i

$$\mathcal{A}_{5(i)}^{(1)\infty} = t_{10} \cdot F_i^5 - \frac{\pi}{2} \sum_{i \neq j} (h_j \cdot k_m) G_F(\nu_j - \nu_m) (t_8 \cdot F_{i,j}^4) - \pi H_i \cdot K_{[6]} \tag{5.6}$$

where H_i is defined as in eq. (2.8),

$$H_i = \sum_{i \neq j} h_j t_8 F_{i,j}^4 \tag{5.7}$$

and $K_{[6]}$ is the total momentum defined in eq. (2.9). The only pieces that could lead to triangles at six-point arise from the contributions

$$\begin{aligned}
 \delta \mathcal{A}_6^{(1)\infty} &= \sum_{i,m=1}^6 h_i \cdot k_m (\nu_i - \nu_m) \mathcal{A}_{5(i)}^{(1)\infty} \\
 &= - \left(\sum_{i=1}^6 h_i [t_{10} \cdot F_i^5 - \frac{\pi}{2} \sum_{i \neq j} (h_j \cdot k_m) G_F(\nu_j - \nu_m) (t_8 \cdot F_{i,j}^4)] \right) \cdot K_{[6]} \\
 &+ \pi \sum_{i=1}^6 (h_i \cdot K_{[6]}) (H_i \cdot K_{[6]})
 \end{aligned} \tag{5.8}$$

The sum over the polarisations can be decomposed on a basis of independent momenta (as in eq. (4.1)) as

$$\sum_{i=1}^6 h_i t_i = \begin{cases} \sum_{i=1}^4 c_i k_i & \text{for } D = 4 \\ \sum_{i=1}^5 c_i k_i + q_{\perp} & \text{for } D \geq 5 \end{cases} \tag{5.9}$$

where the coefficients c_i are constants and t_i is either the combination multiplying h_i in (5.8) or H_i . The constants for each tensorial structure do not have to be identical. These expressions lead to cubic hexagons $I_6^{(D)}[(k_i \cdot K_{[6]})(k_j \cdot K_{[6]})(k_l \cdot K_{[6]})]$ or quartic hexagons $I_6^{(D)}[(k_i \cdot K_{[6]})(k_j \cdot K_{[6]})(k_l \cdot K_{[6]})(k_m \cdot K_{[6]})]$ that will have to be cancelled by implementing the reductions formulas for the six-point integrals [37].

6. Discussion

In this paper we have explored the one-loop n -graviton amplitude derived in the field theory limit ($\alpha' \rightarrow 0$) of type IIA and IIB string theory while preserving maximal supersymmetry of the theory.

In this ‘string based’ formalism the integrand of the one-loop amplitudes in super-gravity takes the form of the square of corresponding super-Yang-Mills amplitudes plus an additional contribution from the zero modes of the bosonic coordinate coupling the left and right moving sectors. We have shown that triangle integral functions are not present at one-loop, in accordance with the “no-triangle hypothesis.”

The Kawai, Lewellen and Tye relations [8], which are derived from string theory, express gravity tree amplitudes as the sums of products of two Yang-Mills tree amplitudes and have many exciting and surprising consequences. Tree-level gravity amplitudes were shown in [26] to enjoy enhanced symmetries similar to those found for Yang-Mills theories inherited via the KLT relations. The surprising good high-energy behaviour of gravity tree amplitudes [43, 27, 44] and the cancellations of certain tree-graphs has been linked to the cancellations of integral functions at one-loop level in [28, 29]. The good high-energy behaviour of gravity amplitudes at tree level was recently attributed to basic gauge symmetry of the underlying gravitational Lagrangian [45]. Gauge invariance was first linked to unexpected cancellations for loop and tree amplitudes in gravity theories in ref. [29].

In this paper we have investigated the cancellations of integral functions at one-loop level encapsulated by the “no-triangle hypothesis”, from the viewpoint of the field theory limit of string theory. We would like to emphasise at this point that the viewpoint of using string theory as a guideline for calculations in analysing the “no-triangle hypothesis” is very different from that of unitarity methods. Our conclusions however remain the same. The origin of triangle cancellation has in this paper been attributed to the decoupling of the longitudinal modes of string theory in the field theory limit and the summation over all the possible orderings of the external legs due to the absence of colour ordering in gravity amplitudes. In the language of field theory this means that the “no-triangle” cancellations has their roots in the gauge invariance of the theory as it appear to be the case for the tree level amplitude simplifications. The cancellations caused by the identities decoupling the longitudinal modes should also apply to the pure spinor formulation of string perturbation theory [31].

At multi-loop level cancellations such as the ones observed at one-loop level might have the potency to ultimately lead to a ultra-violet finite point-like theory of perturbative gravity in four dimensions as was suggested in [28, 30, 33, 6, 29]. Cancellations of potential UV-divergences at three-loop level was examined in [7] and by explicit computation it was

shown at three-loops that the UV-behaviour of maximal supergravity in $D = 4$ is no worse than that of $\mathcal{N} = 4$ super-Yang-Mills.

It is surprising that gauge invariance appears to be the main driving force for the observed simplifications of gravity tree and loop amplitudes. The full symmetry principle behind these unexpected cancellations appear to have the potency to lead to new ground breaking discoveries regarding the UV-behaviour of perturbative gravity. Further investigations are clearly needed — especially at multi-loop level using the explicit information about the origin of cancellations at tree and loop level.

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A. The n -graviton amplitude at genus one in type II string theory

We compute the n -graviton amplitude at one loop in type IIA/B string theory in ten dimensions. With the following normalisations of the world-sheet action for the type II superstring we have

$$S = \frac{1}{2\pi\alpha'} \int d^2z (\partial x^\mu \bar{\partial} x_\mu + \psi^\mu \bar{\partial} \psi_\mu + \bar{\psi}^\mu \partial \psi_\mu) \tag{A.1}$$

where $\alpha' = \ell_s^2$, and the graviton vertex operator in the $(0, 0)$ -ghost picture is

$$V^{(0,0)} = \frac{\kappa_{(10)}}{\alpha'} : h_{\mu\nu} (\partial x^\mu + i k \cdot \psi \psi^\mu) (\bar{\partial} x^\nu + i k \cdot \bar{\psi} \bar{\psi}^\nu) e^{i k \cdot x} : \tag{A.2}$$

We define $\kappa_{(10)}^2 = 2^6 \pi^7 \alpha'^4$ [38]. The symmetric polarisation tensor $h_{\mu\nu}$ is decomposed as $(h_\mu \bar{h}_\nu + h_\nu \bar{h}_\mu)/2$ where h_μ and \bar{h}_μ are polarisation vectors satisfying the transversality condition $k^\mu h_\mu = 0$ and $k^\mu \bar{h}_\mu = 0$.

The string theory S -matrix for an n -graviton amplitude is expanded as

$$\mathbf{A} = \kappa_{(10)}^{n-2} g_s^n \left(\frac{1}{g_s^2} A^{\text{tree}} + 2\pi A^{\text{genus-1}} + \dots \right) \tag{A.3}$$

A.1 General structure of the amplitude

The general structure of the multi-graviton one-loop amplitude in type IIA/B string theory compactified to D dimensions on a $10 - D$ -torus is given by the sum of the *even/even* and

the *odd/odd* spin structure contribution $A^{\text{genus}-1} = A_n^{e/e} + A_n^{o/o}$. The *even/even* spin structure contribution takes the form [16]

$$A_n^{e/e} = \frac{\kappa_{(10)}^2}{\alpha'^{\frac{D}{2}-n}} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \tau_2^{n-5} \Gamma_{(10-D,10-D)} \prod_{i=1}^{n-1} \int_{\mathcal{T}} \frac{d^2z_i}{\tau_2} \left\langle |\mathcal{A}_n|^2 \prod_{i=1}^n e^{i k \cdot x(z_i)} \right\rangle \quad (\text{A.4})$$

where $z_i = \nu_i^{(1)} + i\tau_2 \nu_i$ with $-1/2 \leq \nu_i^{(1)} \leq 1/2$, $0 \leq \nu_i \leq 1$ and $z_n = \tau$. $\Gamma_{(10-D,10-D)}$ is defined as the lattice sum over the winding modes and Kaluza-Klein states of the type II string compactified on a $10 - D$ -torus. The integrand takes the form [37]

$$\mathcal{A}_n = T_{2n} \cdot F^n + \sum_{i=1}^n h_i \cdot \partial x(z_i) \mathcal{A}_{n-1}(\hat{i}) \quad (\text{A.5})$$

where the \hat{i} denotes that state i is not included. The bosonic contributions $h_i \cdot \partial x(z_i)$ can contract either a plane wave factor leading to $h_i \cdot k_j \langle \partial x(z_i) x(z_k) \rangle = h_i \cdot k_j \partial_{z_i} \ln \chi(z_{ij})$ (with $i \neq j$) or contract a left moving $h_j \cdot \partial x(z_j)$ (with $i \neq j$) leading to $h_i \cdot h_j \partial_{z_i}^2 \ln \chi(z_{ij})$ or a right moving $\bar{h}_j \cdot \bar{\partial} x(z_j)$ (with $i \neq j$) leading to $h_i \cdot \bar{h}_j \partial_{z_i} \bar{\partial}_{z_j} \ln \chi(z_{ij})$. The bosonic propagator is given by

$$\ln \chi(z) = \frac{\pi \tau_2 \nu^2}{2} - \frac{1}{4} \ln \left| \frac{\sin(\pi z)}{\pi} \right|^2 - \sum_{m \geq 1} \left(\frac{q^m}{1 - q^m} \frac{\sin^2(m\pi z)}{m} + c.c. \right) \quad (\text{A.6})$$

where $q = \exp(2i\pi\tau)$.

The contractions between the fermions in the vertex operators are given by $\langle \psi^\mu(z) \psi^\nu(0) \rangle_\alpha = \alpha' \eta^{\mu\nu} S_\alpha(z)$

$$S_\alpha(z) = \frac{\theta_\alpha(z|\tau)}{\theta_\alpha(0|\tau)} \frac{\theta'_1(0|\tau)}{\theta_1(z|\tau)} \quad (\text{A.7})$$

for the even spin structures $\alpha = 2, 3, 4$ and

$$S_1(z|\tau) = \frac{\theta'_1(z|\tau)}{\theta_1(z|\tau)} \quad (\text{A.8})$$

for the odd spin structure. Performing such contractions in eq. (A.5) for $T_{2n} \cdot F^n$ one gets

$$T_{2n} \cdot F^n = \sum_{\substack{\sigma \in S_n \\ \sigma = (c_1) \cdots (c_k)}} \text{tr}(F^{i_{c_1(1)}} \cdots F^{i_{c_1(l_1)}}) \cdots \text{tr}(F^{i_{c_k(n-l_k+1)}} \cdots F^{i_{c_k(n)}}) \\ \times G(z_{\sigma(1)} - z_{\sigma(2)}, \cdots, z_{\sigma(n-1)} - z_{\sigma(n)}) \quad (\text{A.9})$$

Hence $T_{2n} \cdot F^n$ is expressed as the sum of products of traces over the decomposition of the permutations σ of the n indices over a product of cycles c_k of length l_k . Because $\text{tr}(F) = 0$ no cycle of length 1 can occur in the decomposition. The function G is expressed in terms of the fermionic propagators as

$$G(x_1, \dots, x_n|\tau) = \sum_{\alpha=2,3,4} (-1)^{\alpha-1} \frac{\theta_\alpha^4(0|\tau)}{\eta^{12}(\tau)} \prod_{j=1}^n S_\alpha(x_j) \quad (\text{A.10})$$

where $x_1 + \dots + x_n = 0$.

The Jacobi identity insures that $G(x_1, \dots, x_n) = 0$ for $n \leq 3$ and $G(x_1, \dots, x_4) = 1$. In the four-point amplitudes the only cycle decompositions of $\sigma \in S_4$ that contribute are $\sigma = (1234)$ and $\sigma = (12)(34)$ and their cyclic permutations, giving rise to the famous $t_8 F^4$ tensor. Using an extension of the Fay trisequent formula one can explicitly evaluate to all orders the sum over the spin structure [37]. The result for $n = 5$ is given by

$$G(x_1, \dots, x_5) = \sum_{j=1}^5 S_1(x_j) \tag{A.11}$$

with $x_1 + \dots + x_5 = 0$.

The odd spin structure begins to contribute from $n \geq 5$ points onwards and takes the form

$$A_n^{\text{odd}} = \frac{\kappa_{(10)}^2}{\alpha'^{\frac{D}{2}-n}} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \int d^{10}\psi_0 d^{10}\bar{\psi}_0 \tau_2^{n-5} \Gamma_{(10-D, 10-D)} \times \\ \times \prod_{i=1}^{n-1} \int \frac{d^2\mathbf{z}_i}{\tau_2} e^{\psi_0^\mu (ik_i^\mu \theta_i + h_i^\mu) + c.c.} \mathcal{A}_n^{\text{odd}} \prod_{1 \leq i < j \leq n} |\chi(z_i - z_j)|^{-2\alpha' k_i \cdot k_j} \tag{A.12}$$

Here $d^2\mathbf{z} = d^2z d\theta d\bar{\theta}$ is the measure of integration over the positions of the insertion points of the vertex operators in the $N = 1$ world-sheet formalism. The amplitude receives an odd spin structure contribution from $n \geq 5$ [40] in ten dimensions. For a toroidal compactification the number of fermionic zero modes will not depend on the dimension because $\mathcal{N} = 8$ supersymmetries are preserved. The integration over the fermionic zero modes is carried out using the rule

$$\int d^{10}\psi_0 \psi_0^{m_1} \dots \psi_0^{m_{10}} = \alpha'^5 10! \epsilon_{10}^{m_1 \dots m_{10}} \tag{A.13}$$

For a compactification of the loop amplitude on a torus of dimension $10 - D$, we will have D zero modes from the space-time part and $10 - D$ zero modes from the internal fermions along the torus directions. For the case of amplitudes with only graviton vertex operators it is not possible to saturate the fermionic zero modes from the internal directions and the amplitude vanishes in $D < 10$.

A.2 The field theory limit

In this paper we are interested in the low-energy limit $\alpha' \rightarrow 0$ of the n -graviton type II string amplitude in $4 \leq D \leq 10$ dimensions. The limit is achieved as in [14] and leads to the $\mathcal{N} = 8$ supergravity field theory amplitude evaluated in the dimensional regularisation scheme.

Compactified on a $10 - D$ dimensional square torus of typical size R the string amplitude described in the previous section takes the form

$$A_{\text{genus}-1} = \frac{\kappa_{(10)}^2}{\alpha'^{\frac{D}{2}-n}} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \tau_2^{n-5} F_n(\tau) \Gamma_{(10-D, 10-D)} \tag{A.14}$$

where $F_n(\tau)$ is the integrand of the n -graviton amplitude given in eq. (A.4). Because we are interested in the supergravity limit all the winding modes and Kaluza-Klein states will be decoupled by taking the scaling limit $R \rightarrow 0$ and $\alpha'/R \rightarrow 0$ [6, 14] (at one-loop the limit is not affected by the issue raised in [19]) with the result

$$\lim_{\substack{\alpha' \rightarrow 0 \\ R \rightarrow 0, \alpha'/R \rightarrow 0}} \Gamma_{(10-D, 10-D)} \rightarrow R^{5-\frac{D}{2}} \tau_2^{5-\frac{D}{2}} \quad (\text{A.15})$$

In the limit $\alpha' \rightarrow 0$ one has to take the string proper time $\tau_2 \rightarrow \infty$ so that $t = \alpha' \tau_2$ and the positions¹ of the vertex operators $z_i = z_i^{(1)} + i\tau_2 \nu_i$ with $\nu_i \in [0, 1]$ stays finite. As well there are some contributions from colliding several (two or more) vertex operators, leading to the reducible contributions represented in figure⁴ 1.

Because of the vanishing of the one-, two-, and three-point amplitudes in type II superstring and $\mathcal{N} = 8$ supergravities, reducible contributions can only appear from $n \geq 5$ graviton amplitudes and are constructed from boxes and higher point amplitudes.

In the scaling limit $\alpha' \rightarrow 0$, one has to take $\tau_2 \rightarrow \infty$ and $t = \alpha' \tau_2$ finite. The bosonic and fermionic propagators have the following limiting expressions

$$S_1(z) \rightarrow G_F(\nu) \equiv \pi \text{sign}(\nu) \quad (\text{A.16})$$

and

$$\partial_z \ln \chi(z) \rightarrow \dot{G}_B(\nu) \equiv \frac{\pi}{2} (2\nu - \text{sign}(\nu)) \quad (\text{A.17})$$

$$\partial_z^2 \ln \chi(z) \rightarrow \ddot{G}_B(\nu) \equiv -\frac{\alpha' \pi}{4t} \quad (\text{A.18})$$

$$\partial_z \bar{\partial}_{\bar{z}} \ln \chi(z) \rightarrow \ddot{G}_B(\nu) \equiv -\frac{\alpha' \pi}{4t} \quad (\text{A.19})$$

Because of the zero mode contributions from the coordinates $x^\mu(z)$ the second derivative of the bosonic propagator contribute to an inverse power of the proper time. This leads to a shift in the dimension of the resulting field theory loop amplitude.

In this limit the fermionic contractions $T_{2n} \cdot F^n$ in eq. (A.9) leads to

$$\begin{aligned} t_{2n} \cdot F^n &\equiv \lim_{\alpha' \rightarrow 0} T_{2n} \cdot F^n \\ &= \sum_{\sigma \in S_n} \text{tr}(F^{i_{c_1(1)}} \dots F^{i_{c_1(l_1)}}) \dots \text{tr}(F^{i_{c_k(n-l_k+1)}} \dots F^{i_{c_k(n)}}) \\ &\quad \times G^\infty(z_{\sigma(1)} - z_{\sigma(2)}, \dots, z_{\sigma(n-1)} - z_{\sigma(n)}) \end{aligned} \quad (\text{A.20})$$

where $G^\infty(x_1, \dots, x_n) = 0$ for $n \leq 3$, $G^\infty(x_1, \dots, x_4) = 1$ and

$$G^\infty(x_1, \dots, x_5) = \sum_{i=1}^5 \text{sign}(x_i) \quad (\text{A.21})$$

¹The differences between the ν_i give the Feynman parameters of the field theory loop amplitude once an ordering of the external leg has been chosen.

In the field theory limit the factor from the contractions between the plane waves approximates to

$$\left\langle \prod_{1 \leq i < j \leq n} e^{ik_i \cdot x(z_i)} \right\rangle = \prod_{1 \leq i < j \leq n} \chi(z_i - z_j)^{-\alpha' k_i \cdot k_j} \rightarrow \prod_{1 \leq i < j \leq n} \exp(-\pi t Q_n) \quad (\text{A.22})$$

with Q_n defined in eq. (2.5). Depending on the number of first and second derivatives of the bosonic propagators and the number of fermionic propagators, one gets that the field theory one-loop integrals are given by

$$\begin{aligned} M_n &= \int_0^1 \frac{dt}{t} t^{m+n-\frac{D}{2}} \prod_{i=1}^{n-1} \int_0^1 d\nu_i \nu_1 \cdots \nu_k e^{-\pi t Q_n} \\ &= \pi^{\frac{D}{2}-m-n} \Gamma\left(m+n-\frac{D}{2}\right) \prod_{i=1}^{n-1} \int_0^1 d\nu_i \nu_1 \cdots \nu_k Q_n^{\frac{D}{2}-m-n} \end{aligned} \quad (\text{A.23})$$

This is the expression for the n -point integrals $I_n^{(D+2m)}[\nu_1 \cdots \nu_n]$ summed over all orderings of the external legs.

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